

An Example of non-quenched Convergence in the Conditional CLT for Discrete Fourier Transforms

David Barrera

Department of Mathematical Sciences
University of Cincinnati

PO Box 210025, Cincinnati, Oh 45221-0025, USA.

Email: barrerjd@mail.uc.edu

Abstract

A recent result by Barrera and Peligrad ([1], Theorem 1) shows that the quenched Central Limit Theorem holds for the components of the discrete Fourier transforms of a stationary process in L^2 orthogonal to the subspace of functions that are measurable with respect to the initial sigma-field. In this note we address the question of whether the quenched CLT remains true for the Fourier transforms without taking orthogonal projections, as could be expected in view of previous, related results about the annealed convergence of the process under consideration (see for instance [6], Theorem 2.1).

We give a negative answer to this question by exhibiting an example of a process satisfying the hypothesis of Theorem 1 in [1] for which the Fourier transforms do not satisfy a quenched limit theorem. The proof combines ideas from a construction due to Volný and Woodroffe (see [8]) with an interpretation of the results in [1] in the case of linear processes, and with applications of some previous results related to discrete Fourier transforms.

MSC 2010 subject classification: 60F05 60G42 60G48 60G10 42A16 42A55 42A61 .

Keywords: Discrete Fourier transform, central limit theorem, martingale approximation, quenched convergence.

1 Introduction and Notation.

Let $(X_n)_{n \in \mathbb{Z}}$ be a strictly stationary centered, ergodic sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This is: $X_n = X_0 \circ T^n$, where $T : \Omega \rightarrow \Omega$ is an ergodic, bimeasurable, invertible transformation and $EX_0 = 0$. Assume that $X_0 \in L^2_{\mathbb{P}}(\mathcal{F}_0)$ where $\mathcal{F}_0 \subset \mathcal{F}$ is a sigma algebra satisfying $\mathcal{F}_0 \subset T^{-1}\mathcal{F}_0$ (i.e. T^{-1} is \mathcal{F}_0 measurable), define $\mathcal{F}_n := T^{-n}\mathcal{F}_0$ for all $n \in \mathbb{Z}$ and $\mathcal{F}_{-\infty} := \cap_{n \in \mathbb{Z}} \mathcal{F}_n$. Denote by E_n the conditional expectation with respect to \mathcal{F}_n . So $E_n Z := E[Z|\mathcal{F}_n]$ for every integrable random variable Z .

Define, for every $\theta \in [0, 2\pi)$ the n -th discrete Fourier transform of $(X_k(\omega))_k$ at θ by

$$S_n(\theta, \omega) := \sum_{k=0}^{n-1} e^{ik\theta} X_k(\omega). \quad (1)$$

When $\theta \in (0, 2\pi)$ is fixed, we will denote by $S_n(\theta)$ the random variable $S_n(\theta, \cdot)$. In the special case $\theta = 0$ we denote by S_n the random variable $S_n(0, \cdot)$. So $S_n(\omega) := \sum_{k=0}^{n-1} X_k(\omega)$.

Assume also that E_0 is regular. This is, that there exists a family of measures $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ such that for every integrable function X ,

$$\omega \mapsto \int_{\Omega} X(\omega') d\mathbb{P}_\omega(\omega')$$

defines a version of $E_0 X$.

Finally, denote by λ the Lebesgue measure on $[0, 2\pi)$ (or any other Borelian in \mathbb{R}).

Under these assumptions Barrera and Peligrad, in [1], proved the following theorem:

Theorem 1. *There exists a set $I \subset (0, 2\pi)$ with $\lambda(I) = 2\pi$ such that, for all $\theta \in I$, the complex-valued random variables*

$$Y_n(\theta) := \frac{1}{\sqrt{n}}(S_n(\theta) - E_0 S_n(\theta)) \quad (2)$$

converge to a complex Gaussian random variable under \mathbb{P}_ω for all ω in a set Ω_θ with $\mathbb{P}(\Omega_\theta) = 1$. The asymptotic distribution of the real and imaginary parts corresponds to a bivariate Gaussian random variable with independent entries, each with mean zero and variance

$$\sigma_\theta^2 = \lim_{n \rightarrow \infty} \frac{E_0 |S_n(\theta) - E_0 S_n(\theta)|^2}{2n}.$$

(the limit exists with probability one, and it is nonrandom).

2 Quenched Convergence

In the context of the previous section, and given a distribution function F_Y (associated to some random variable Y defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$), we say that the process Y_n converges to Y in the quenched sense, denoted here by $Y_n \Rightarrow_Q Y$, if for almost every ω , and every continuous and bounded function f , $E^\omega f(Y_n) \rightarrow_{n \rightarrow \infty} E f(Y)$, where E^ω denotes integration with respect to \mathbb{P}_ω and $E f(Y) := \int_{\mathbb{R}} f(y) dF_Y(y) = \int_{\Omega'} f \circ Y(\omega') d\mathbb{P}'(\omega')$.

In other words, we require that $E_0 f(Y_n) \rightarrow E f(Y)$ a.s. (over a set not depending of f). By Portmanteau's theorem, this amounts to say that for almost every ω , $\mathbb{P}_\omega(Y_n \leq y) \rightarrow F_Y(y)$ at every continuity point y of F_Y . Theorem 1 is thus a statement about the quenched convergence of $Y_n(\theta)$ for every $\theta \in I$.

Note that quenched convergence implies convergence in distribution ("annealed" convergence) by the dominated convergence theorem: for every uniformly continuous and bounded function f ,

$$\int_{\Omega} f(Y_n) d\mathbb{P}(\omega) = \int_{\Omega} E_0 f(Y_n)(\omega) d\mathbb{P}(\omega) \xrightarrow{n \rightarrow \infty} \int_{\Omega} E f(Y) d\mathbb{P}(\omega) = E f(Y).$$

In particular, Theorem 1 relates to some previous results about annealed convergence (see for instance [6] and the references therein).

It is worth to remark that, without additional assumptions, the methods of [1] do not give a description of the elements in the set I . The martingale version of the theorem, used to approximate the general case, works provided that e^{-2it} is not an eigenvalue of the Koopman operator associated to T (namely $f \mapsto f \circ T$ for all $f \in L^2_{\mathbb{P}}(\Omega, \mathbb{C})$), and therefore we consider these as exceptional values. A consideration of the classical case $\theta = 0$ shows that more hypotheses may be needed to guarantee the convergence in distribution of $Y_n(\theta)$ outside of I .

2.1 Possible limit Laws for a given initial point.

Suppose that we know of an integrable process $(Y_n)_n$ that $Y_n - E_0 Y_n \Rightarrow_Q Y$, say $Y_n - E_0 Y_n \Rightarrow Y$ under \mathbb{P}_ω for $\omega \in \Omega_0$, where $\mathbb{P}(\Omega_0) = 1$. What are the possible limit laws for Y under \mathbb{P}_ω , for a fixed ω ?

To answer this question we depart from the following result (see the proof of Lemma 18 in [1]):

Lemma 1. *If $\mathcal{F}_0 \subset \mathcal{F}$ is a σ -algebra for which $E[\cdot | \mathcal{F}_0] =: E_0$ admits a regular version in the sense explained above (see the introduction), Y is \mathcal{F}_0 -measurable, and X is a given random variable, then there exists $\Omega_{1,Y} \subset \Omega$ with $\mathbb{P}(\Omega_Y) = 1$ such that, for every $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ continuous and bounded*

$$E^\omega[g(X, Y(\omega))] = E^\omega[g(X, Y)] \quad (3)$$

for all $\omega \in \Omega_Y$.

This lemma, in combination with Proposition 4 and Lemma 7 in the appendix, gives the following proposition.

Proposition 1 (Possible limit Laws for a fixed starting point.). *With the notation of Lemma 1, assume that $(Y_n)_n$ is an integrable process such that $Y_n - E_0 Y_n \Rightarrow Y$ under \mathbb{P}_ω for all $\omega \in \Omega_0$ ($\Omega_0 \subset \Omega$ is any given set, not even assumed measurable), and let $\Omega_1 := \cap_n \Omega_{Y_n}$. Then given $\omega \in \Omega_0 \cap \Omega_1$, Y_n is convergent under \mathbb{P}_ω to some random variable Z_ω if and only if $L(\omega) = \lim_{n \rightarrow \infty} E_0 Y_n(\omega)$ exists, in which case $Z_\omega = Y + L(\omega)$ (in distribution).*

Proof: Given $\omega \in \Omega_0 \cap \Omega_1$ and any bounded and continuous function g

$$E^\omega g(Y_n - E_0 Y_n(\omega)) = E^\omega g(Y_n - E_0 Y_n) \rightarrow_n E g(Y),$$

so that $Y_n - E_0 Y_n(\omega) \Rightarrow Y$ under \mathbb{P}_ω . From $Y_n = Y_n - E_0 Y_n(\omega) + E_0 Y_n(\omega)$ the conclusion follows via Proposition 4 if Y is not constant and via Lemma 7 if Y is constant. \square

2.2 The Question

Let us define, for every $\theta \in [0, 2\pi)$, $Z_n(\theta) := \frac{1}{\sqrt{n}} S_n(\theta)$. It is natural to ask whether the random centering in (2) is necessary to obtain quenched convergence. For regular processes (namely $E[X_0 | \mathcal{F}_{-\infty}] = 0$) this amounts, in view of Theorem 2.1 in [6], to whether the conclusion of Theorem 1 holds with $Z_n(\theta)$ in place of $Y_n(\theta)$.

The first observation in this direction is given by the following lemma.

Lemma 2. *With the notation of Theorem 1, fix $\theta \in (0, 2\pi)$ (θ may or may not be in I), and assume that $Y_n(\theta) \Rightarrow_Q Y_\theta$ for some Y_θ . Then $Z_n(\theta) \Rightarrow_Q Y_\theta$ if and only if $E_0 S_n(\theta) = o(\sqrt{n})$ almost surely.*

Proof: *Sufficiency.* First note that for any set $A \in \mathcal{F}$, $\mathbb{P}(A) = \int_A \mathbb{P}_\omega(A) d\mathbb{P}(\omega)$. Applying this observation to the complement of $[E_0 S_n(\theta) = o(\sqrt{n})]$ we see that if $E_0 S_n(\theta) = o(\sqrt{n})$ a.s. then $E_0 S_n(\theta) = o(\sqrt{n})$ \mathbb{P}_ω -a.s. for \mathbb{P} -a.e. ω , and therefore the asymptotic distributions of $Y_n(\theta)$ and $Z_n(\theta)$ (if any) must be the same under \mathbb{P}_ω for \mathbb{P} -a.e. ω . This proves sufficiency.

Necessity. We appeal to Proposition 1 (applied to the real and imaginary parts of the processes in question) by taking, in place of Ω_0 , $\Omega_\theta := \{\omega \in \Omega : Y_n(\theta) \Rightarrow Y_\theta \text{ under } \mathbb{P}_\omega\}$. This gives that $Y_\theta = Y_\theta + \lim_n E_0 S_n(\theta, \omega) / \sqrt{n}$ for $\omega \in \Omega_{1,\theta} := \Omega_1 \cap \Omega_\theta$. This is, that

$$\lim_n \frac{E_0 S_n(\theta, \omega)}{\sqrt{n}} = 0 \quad \text{for all } \omega \in \Omega_{1,\theta}.$$

\square

Therefore, to give a negative answer to our question, we must provide a regular process $(X_n)_n$ satisfying the hypothesis of Theorem 1 for which

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n}} E_0 S_n(\theta) \right| > 0 \right) > 0 \quad \text{for } \theta \text{ in a set } I' \text{ with } \lambda(I') > 0. \quad (4)$$

Proving (4) gives, in particular, the necessity of random centering for a nonnegligible subset of I (namely $I \cap I'$).

In their paper [8], Volny and Woodroffe provide an example of a sequence $(X_n)_n$ for which a quenched CLT holds for $(Y_n(0))_n$ but not for $(Z_n(0))_n$. In this paper, we adapt their construction to give an example satisfying (4) with $I' = [0, 2\pi)$. The random centering “ $-E_0 S_n(\theta)$ ” is therefore a necessary condition for the conclusion of Theorem 1 to hold.

The main novelty adapting the example in [8], which arises from a careful construction of a sequence $(a_n)_n$ of nonnegative coefficients of a linear process is that, in order to guarantee the validity of the “inductive step” defining a_{n+1} from a_1, \dots, a_n , one needs to prove that a certain type of convergence is uniform in θ (see Lemma 3 below). While it would be sufficient to prove this uniform convergence for θ in an open subinterval I' of $[0, 2\pi)$ in order to construct a valid example, a compactness argument allows us to do it for $I' = [0, 2\pi)$.

We will give an example of a process which, indeed, has the following property

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n}} E_0 S_n(\theta) \right| = \infty \right) = 1 \quad \text{for all } \theta \in [0, 2\pi). \quad (5)$$

For this process, Proposition 1 shows that $\frac{1}{\sqrt{n}} S_n(\theta)$ cannot admit an asymptotic limit under \mathbb{P}_ω for \mathbb{P} -a.e ω .

The rest of the paper is presented as follows: in Section 3 we specialize our study to the case in which $(X_k)_{k \in \mathbb{Z}}$ is a linear process generated by convolution of a sequence of i.i.d random variables and a sequence in $l^2(\mathbb{N})$. For this family of processes, we give an interpretation of the results above in terms of convergence of Fourier series of perodic functions (Proposition 2), and introduce a result necessary to construct the example (Lemma 3). In Section 4 we present the construction itself.

3 The Fourier Transforms of a Linear Process

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $l^2(\mathbb{N})$ (namely $\sum_n a_n^2 < \infty$), and let $(\xi_k)_{k \in \mathbb{Z}}$ be an i.i.d. sequence of centered, square integrable random variables defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. The *linear process* $(X_k)_{k \in \mathbb{Z}}$ generated by (a_k) and (ξ_k) is defined by

$$X_k := \sum_{j \in \mathbb{N}} a_j \xi_{k-j}. \quad (6)$$

The orthogonality of $(\xi_k)_k$ shows that X_k is well defined as an element of $L^2_{\mathbb{P}}$, and that $EX_k^2 = \|\xi_0\|_2^2 \sum_j a_j^2$.

We can regard $\Omega = \mathbb{R}^{\mathbb{Z}}$ as a probability space whose sigma algebra is the product sigma algebra and whose probability measure is $\mathbb{P} = \mathbb{P}' \xi^{-1}$, where $\xi : \Omega' \rightarrow \Omega$ is given by $\xi(\omega') := (\xi_j(\omega'))_{j \in \mathbb{Z}}$. It is well known that, with this structure (because $(\xi_k)_k$ is i.i.d.), the left shift $T : \Omega \rightarrow \Omega$ characterized by $x_k \circ T = x_{k+1}$, where $x_j : \Omega \rightarrow \mathbb{R}$ is the projection on the j -th coordinate, is weakly mixing (and therefore also ergodic).

Note that the coordinate functions x_j are a “copy” of the sequence (ξ_j) : they are independent and have the same distribution. In particular, X_k can be regarded as the function $X_k : \Omega \rightarrow \mathbb{R}$ given by $X_k(\omega) = X_k((x_j(\omega))_j) := \sum_j a_j x_{k-j}(\omega)$.

Clearly, $X_k = X_0 \circ T^k$. In this way $(X_k)_{k \in \mathbb{Z}}$ can be interpreted as a strictly stationary, centered, and ergodic sequence in $L^2_{\mathbb{P}}$.

In this case, we choose $\mathcal{F}_n := \sigma((x_k)_{k \leq n})$ for all $n \in \mathbb{Z}$ and we define \mathbb{P}_{ω} as the measure corresponding to “partial integration with respect to the future”. This is: given $\omega_0 \in \Omega$, $\mathbb{P}_{\omega_0} = \mathbb{P}\pi_{\omega_0}^{-1}$ where $\pi_{\omega_0} : S \rightarrow S$ is given by

$$x_k(\pi_{\omega_0}(\omega)) = \begin{cases} x_k(\omega_0) & \text{if } k \leq 0 \\ x_k(\omega) & \text{if } k > 0 \end{cases}$$

This brings us to the hypotheses at the beginning. By Kolmogorov’s zero-one law, $(X_n)_n$ is a regular process.

As T is weakly mixing its only eigenvalue is $\lambda_0 = 1$ (see for instance [7], p.65).

Coefficients of the Fourier Transforms

Under the given hypothesis $((a_n)_n \in l^2(\mathbb{N}))$, Carleson’s theorem ([3]) guarantees the convergence a.s. of the series

$$f(\theta) = \sum_{j \geq 0} a_j e^{ij\theta} \quad (7)$$

and $f(\theta)$, thus defined, is a 2π -periodic function, square integrable over $[0, 2\pi)$, and satisfying $\hat{f}(n) = a_n$, where \hat{f} denotes the Fourier transform

$$\hat{f}(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ixy} f(y) d\lambda(y).$$

Denote by $f_k(\theta) := \sum_{j=0}^{k-1} a_j e^{ij\theta}$ ($f_k = 0$ if $k \leq 0$). Then we have the following two expressions for $S_n(\theta)$:

$$S_n(\theta) = \sum_{j=-\infty}^{n-1} (f_{-j+n} - f_{-j})(\theta) \xi_j e^{ij\theta}, \quad (8)$$

$$S_n(\theta) = \sum_{k=0}^{\infty} a_k \sum_{j=0}^{n-1} e^{ij\theta} \xi_{j-k}. \quad (9)$$

Now let’s denote, for all $k \geq 0$,

$$\zeta_{-k}(\theta) := \sum_{j=0}^k e^{-ij\theta} \xi_{-j} \quad (\zeta_{-k} = 0 \text{ if } k < 0). \quad (10)$$

Then from (8) and (9) the following two equalities follow respectively:

$$E_0 S_n(\theta) = \sum_{j \leq 0} \xi_j (f_{-j+n} - f_{-j})(\theta) e^{ij\theta}, \quad (11)$$

$$E_0 S_n(\theta) = \sum_{j \geq 0} a_j (\zeta_{-j} - \zeta_{-j+n})(\theta) e^{ij\theta}. \quad (12)$$

In particular

$$\begin{aligned} E_0 |S_n(\theta) - E_0 S_n(\theta)|^2 &= E_0 \left| \sum_{j=1}^{n-1} e^{ij\theta} \xi_j f_{-j+n}(\theta) \right|^2 = \\ &= \|\xi_0\|_2^2 \sum_{j=1}^{n-1} |f_{n-j}(\theta)|^2 = \|\xi_0\|_2^2 \sum_{j=1}^{n-1} |f_j(\theta)|^2, \end{aligned}$$

so that

$$\lim_{n \rightarrow \infty} \frac{E_0 |S_n(\theta) - E_0 S_n(\theta)|^2}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} \|\xi_0\|_2^2 |f_j(\theta)|^2 = \|\xi_0\|_2^2 |f(\theta)|^2.$$

Using this, we get the following version of Theorem 1.

Proposition 2. *For a linear process (6) and almost every $\theta \in (0, 2\pi)$, (2) is asymptotically normally distributed under \mathbb{P}_ω , for \mathbb{P} -almost every ω , with independent real and imaginary parts, each with mean zero and variance*

$$\sigma_\theta^2 = \frac{\|\xi_0\|_2^2 |f(\theta)|^2}{2},$$

where f is given by (7).

By [4], p.4075 (Section 4.1) applied to the sequence $(\delta_{1j})_{j \in \mathbb{Z}}$ (δ_{ij} denotes the Kronecker δ -function) and the fact that T is weakly mixing, the following Law of the Iterated Logarithm holds: for every $t \in (0, 2\pi) \setminus \{\pi\}$

$$\limsup_{n \rightarrow \infty} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n \log \log n}} = \|\xi_0\|_2. \quad (13)$$

almost surely (note that, for the linear process $(\xi_n)_{n \in \mathbb{Z}}$, corresponding to convolution with $(\delta_{1j})_{j \in \mathbb{Z}}$, the spectral density with respect to Lebesgue measure is the constant function $\|\xi_0\|_2^2/2\pi$).

If $\theta = 0$ or $\theta = \pi$, the L.I.L. as stated above holds with $\|\xi_0\|_2$ replaced by $\sqrt{2}\|\xi_0\|_2$ (in this case the process $(\zeta_n(0))_n$ is real-valued).

The equality (13) clearly implies that $\limsup_n \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} \rightarrow \infty$ a.s. The following lemma states that the divergence occurs “at comparable speeds” for every θ .

Lemma 3. Consider the linear process (6) and define ζ_{-k} as in (10). Then for every $\lambda \in \mathbb{R}$ and every $0 < \eta \leq 1$ there exists an $N \in \mathbb{N}$ satisfying

$$\mathbb{P} \left(\max_{1 \leq n \leq N} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} > \lambda \right) \geq 1 - \eta$$

for all $\theta \in [0, 2\pi]$. In particular

$$\mathbb{P} \left(\max_{1 \leq n \leq m} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} \geq \lambda \right) \geq 1 - \eta$$

for all $m \geq N$.

Proof: Fix $\lambda \in \mathbb{R}$ and $0 < \eta \leq 1$. Let¹ $\theta \in [0, 2\pi]$ and $\epsilon > 0$ be given and define

$$E_{\epsilon, m}(\theta) := \left[\inf_{|\delta| < \epsilon} \left\{ \max_{1 \leq n \leq m} \frac{|\zeta_{-n}(\theta + \delta)|}{\sqrt{n}} \right\} > \lambda \right]$$

and

$$E_m(\theta) := \left[\max_{1 \leq n \leq m} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} > \lambda \right].$$

Note that, for fixed m , the sequence of sets $E_{\epsilon, m}(\theta)$ is decreasing with respect to ϵ ($\epsilon_1 < \epsilon_2$ implies that $E_{\epsilon_2, m}(\theta) \subset E_{\epsilon_1, m}(\theta)$), and that the (random) function $\theta \mapsto \max_{1 \leq n \leq m} |\zeta_{-n}(\theta)|/\sqrt{n}$ is continuous for all m . In particular

$$\bigcup_{\epsilon > 0} E_{\epsilon, m}(\theta) = E_m(\theta), \quad (14)$$

where the union is increasing over ϵ decreasing to 0.

By (13), there exists a minimal $N(\theta)$ such that $\mathbb{P}(E_{N(\theta)}(\theta)) > 1 - \eta$ (to see this note that the family $\{E_k(\theta)\}_{k \geq 0}$ is increasing with k , and its union contains the set $[\limsup_n |\zeta_{-n}(\theta)|/\sqrt{n} > \lambda]$, which has measure 1 by (13)) and therefore, by (14), there exists an ϵ_θ such that

$$\mathbb{P}(E_{\epsilon_\theta, N(\theta)}(\theta)) > 1 - \eta. \quad (15)$$

Now, the family of sets $\{(\theta - \epsilon_\theta, \theta + \epsilon_\theta)\}_{\theta \in [0, 2\pi]}$ is an open cover of $[0, 2\pi]$, and therefore it admits an open subcover $\{(\theta_j - \epsilon_j, \theta_j + \epsilon_j)\}_{j=1}^r$ (where $\epsilon_j := \epsilon_{\theta_j}$).

Let $N = \max\{N(\theta_1), \dots, N(\theta_r)\}$. We claim that, for every $\theta \in [0, 2\pi]$

$$\mathbb{P}(E_N(\theta)) > 1 - \eta.$$

Indeed, given $\theta \in [0, 2\pi]$, with $\theta_j - \epsilon_j < \theta < \theta_j + \epsilon_j$,

¹We work over the interval $[0, 2\pi]$ (instead of $[0, 2\pi)$). This has no effect for the validity of the conclusion and is assumed in order to take advantage of compactness, as will be clear along the proof.

$$E_N(\theta) \supset E_{N(\theta_j)}(\theta) = \left[\max_{1 \leq n \leq N(\theta_j)} \frac{|\zeta_{-n}(\theta_j + \theta - \theta_j)|}{\sqrt{n}} > \lambda \right] \supset E_{\epsilon_j, N(\theta_j)}(\theta_j),$$

and the conclusion follows from (15) and the definition of $E_N(\theta)$. \square

4 The Example

We now proceed to prove, by an explicit construction, the following proposition:

Proposition 3. *There exists a square summable sequence $(a_n)_n$ such that, if $(X_n)_n$ is defined by (6) and $S_n(\theta)$ is defined by (1), then*

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} = \infty \right) = 1$$

for all $\theta \in [0, 2\pi)$.

Before giving the proof we depart from the following observation: if $(n_k)_{k \geq 0}$ is a strictly increasing sequence of natural numbers and if $(a_j)_j$ is square summable and satisfies $a_j = 0$ if $j \notin \{n_k\}_k$ then, using (12)

$$\begin{aligned} E_0 S_n(\theta) &= \sum_{j=0}^k e^{in_j \theta} a_{n_j} (\zeta_{-n_j} - \zeta_{-n_j+n})(\theta) + \sum_{j=k+1}^{\infty} e^{in_j \theta} a_{n_j} (\zeta_{-n_j} - \zeta_{-n_j+n})(\theta) =: \\ &A_k(n, \theta) + B_k(n, \theta) \end{aligned} \quad (16)$$

so that

$$\begin{aligned} &\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \right) \geq \\ &\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|A_k(n, \theta)|}{\sqrt{n}} \geq 2^{k+1} \right) - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|B_k(n, \theta)|}{\sqrt{n}} \geq 2^k \right) \geq \\ &\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|A_k(n, \theta)|}{\sqrt{n}} \geq 2^{k+1} \right) - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right). \end{aligned} \quad (17)$$

Now, if $n_{k-1} < n \leq n_k$ then, actually

$$A_k(n, \theta) = \sum_{j=0}^{k-1} e^{in_j \theta} a_{n_j} \zeta_{-n_j}(\theta) + e^{in_k \theta} a_{n_k} (\zeta_{-n_k} - \zeta_{-n_k+n})(\theta).$$

The first summand at the right hand side in this expression is bounded by

$$\sum_{j=1}^{k-1} \sum_{r=0}^{n_j} |a_{n_j}| |\xi_{-r}|$$

and therefore there exists $\lambda_k > 0$ such that

$$\mathbb{P} \left(\left| \sum_{j=0}^{k-1} e^{in_j \theta} a_{n_j} \zeta_{-n_j}(\theta) \right| > \lambda_k \right) \leq \left(\frac{1}{2} \right)^{k+2} \quad (18)$$

for all $\theta \in [0, 2\pi]$.

All together (16), (17) and (18) give the following result.

Lemma 4. *Let $(n_k)_k$ be a strictly increasing sequence of natural numbers and let $(a_j)_j$ be a square summable sequence with $a_j = 0$ for $j \notin \{n_k\}_k$. Then for every sequence of real numbers $(\lambda_k)_k$ satisfying (18) the following inequality holds*

$$\begin{aligned} \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \right) &\geq \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|(\zeta_{-n_k} - \zeta_{-n_k+n})(\theta)|}{\sqrt{n}} \geq \frac{\lambda_k + 2^{k+1}}{a_{n_k}} \right) \\ &\quad - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right) - \left(\frac{1}{2} \right)^{k+2} \end{aligned} \quad (19)$$

for all $\theta \in [0, 2\pi]$.

This, together with the previous lemmas, gives the pieces to construct the example stated in Proposition 3.

Proof of Proposition 3: Following [8], assume that $\|\xi_0\|_2 = 1$ and let $(n_j)_{j \geq 0}$, $(a_j)_{j \geq 0}$, and $(\lambda_j)_{j \geq 0}$ be defined inductively as follows: $n_0 = 1$, $\lambda_0 = 0$, $a_0 = 0$, $a_1 = \frac{1}{2}$, and given n_0, \dots, n_{k-1} , $a_0, \dots, a_{n_{k-1}}$ and $\lambda_0, \dots, \lambda_{k-1}$, let λ_k be such that

$$\mathbb{P} \left(\left| \sum_{j=1}^{k-1} a_{n_j} e^{in_j \theta} \zeta_{-n_j}(\theta) \right| > \lambda_k \right) \leq \left(\frac{1}{2} \right)^{k+2},$$

(see(18)) and let $n_k > n_{k-1}$ be such that

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|(\zeta_{-n_{k-1}} - \zeta_{-n_{k-1}+n})(\theta)|}{\sqrt{n}} \geq \frac{\lambda_k + 2^{k+1}}{a_{n_{k-1}}} \right) \geq 1 - \left(\frac{1}{2} \right)^{k+1} \quad (20)$$

for all $\theta \in [0, 2\pi]$. The choice of n_k is possible according to Lemma 3 ($|(\zeta_{-n_{k-1}} - \zeta_{-n_{k-1}+n})(\theta)|$ and $|\zeta_{-n}(\theta)|$ have the same distribution). Then define $a_{n_k} = \frac{1}{2^k \sqrt{n_{k-1}}}$ and $a_j = 0$ for $n_{k-1} < j < n_k - 1$.

The sequences $(a_j)_{j \geq 0}$ and $(\lambda_k)_k$, thus defined, satisfy the hypotheses of Lemma 4 and therefore, by the estimates (19) and (20),

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \right) \geq 1 - \left(\frac{1}{2} \right)^{k+2} - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right)$$

for all $\theta \in [0, 2\pi]$.

Now we show that, under the present conditions,

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right) \leq \left(\frac{1}{2} \right)^{k+2} \quad (21)$$

for $k \geq 3$.

Fix $k \geq 3$. First we recall the following (Doob's) maximal inequality for martingales (see [5]): if $(M_n)_n$ is a L^p submartingale (namely $\|M_n\|_p := (E|M_n|^p)^{1/p} < \infty$ for all n) for some $p > 1$ then

$$\left\| \sup_{k \leq n} M_k \right\|_p \leq \frac{p}{p-1} \|M_n\|_p. \quad (22)$$

Now, for fixed θ , $(|\zeta_{-n}(\theta)|)_{n \geq 0}$ is an L^2 submartingale (with respect to $(\mathcal{G}_n)_n$, where $\mathcal{G}_k = \sigma((\xi_{-j})_{j \leq k-1})$) and therefore, by Doob's maximal inequality (22):

$$E \left(\max_{k \leq n} |\zeta_{-k}(\theta)| \right) \leq \left\| \max_{k \leq n} |\zeta_{-k}(\theta)| \right\|_2 \leq 2 \|\zeta_{-n}(\theta)\|_2 \leq 2\sqrt{n}.$$

This gives

$$\begin{aligned} E \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \right) &\leq \sum_{j=k+1}^{\infty} a_{n_j} E \left(\max_{k \leq n_k - n_{k-1}} |\zeta_{-k}(\theta)| \right) \leq \\ &\sum_{j=k+1}^{\infty} \frac{1}{2^{j-1}} \sqrt{\frac{n_k - n_{k-1}}{n_{j-1}}} \leq \frac{1}{2^{k-1}}, \end{aligned}$$

and therefore, by Markov's inequality

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right) \leq \frac{1}{2^{2k-1}} \leq \left(\frac{1}{2} \right)^{k+2}$$

as claimed.

The proof is finished as follows: a combination of (19), (20) and (21) gives, under the present choices of $(a_k)_k$ and $(n_k)_k$, that

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} < 2^k \right) \leq \left(\frac{1}{2} \right)^{k+1}$$

so that, by the first Borel-Cantelli Lemma

$$\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \text{ except for finitely many } k \text{'s,}$$

\mathbb{P} -a.s. This clearly implies that $\limsup_n |E_0 S_n(\theta)|/\sqrt{n} = \infty$ \mathbb{P} -a.s. \square

Appendix: convergence of Types

A distribution function F is *non-degenerate* if it is not a Heaviside function (this is, if it is not the indicator function of some interval $[a, +\infty)$). We recall the following *Convergence of Types* theorem ([2], Th 14.2).

Lemma 5. *Let F_n , F and G be distribution functions, and a_n, u_n, b_n, v_n be constants with $a_n > 0, u_n > 0$. If F, G are nondegenerate, $F_n(a_n x + b_n) \Rightarrow F(x)$, and $F_n(u_n x + v_n) \Rightarrow G(x)$ then there exist $a = \lim_n a_n/u_n$, $b = \lim_n (b_n - v_n)/u_n$ and $G(x) = F(ax + b)$.*

Note that, necessarily, $a > 0$ (as otherwise G would be constant).

We will translate this statement to a statement about convergence of stochastic processes (with a restricted choice of u_n, v_n , see Proposition 4 below). To begin, we remind the following elementary facts, here \rightarrow_P denotes convergence in probability.

1. If a is constant then $U_n \Rightarrow a$ if and only if $U_n \rightarrow_P a$.
2. If $U_n \Rightarrow W$ and $V_n \rightarrow_P 0$ then $U_n + V_n \Rightarrow W$.
3. If $(a_n)_n$ is a sequence of constant functions then $a_n \Rightarrow A$ if and only if $a = \lim_n a_n$ exists (and therefore $A = a$ a.s.).

Lemma 6. *If $Y_k \Rightarrow Y$ and $\{c_k\}_k \subset \mathbb{R}$ are such that $Y_k + c_k \Rightarrow 0$, then $Y = -\lim_k c_k$. In particular, Y is a constant function.*

Proof: Note that $c_k = -Y_k + (Y_k + c_k) \Rightarrow -Y$ because $Y_k + c_k \Rightarrow 0$. Now use 2. and 3. above. \square

Corollary 2. *If X is not constant, $X_n \Rightarrow X$, and a_n, b_n are such that $a_n X_n + b_n \Rightarrow 0$, then $a_n \rightarrow 0$ and $b_n \rightarrow 0$.*

Proof: If $0 < a := \limsup_n a_n \leq \infty$ and $a_{n_k} \rightarrow_{k \rightarrow \infty} a$ with $a_{n_k} > 0$, then applying Lemma 6 with $Y_k = X_{n_k}$ and $c_k = b_{n_k}/a_{n_k}$ we conclude that X is constant. This proves that, necessarily, $\limsup_n a_n \leq 0$. A similar argument shows that $\liminf_n a_n \geq 0$, and therefore $\lim_n a_n = 0$.

The fact that $b_n \rightarrow 0$ follows from here applying Lemma 6 again, because $a_n X_n \Rightarrow 0$. \square

These results give rise to the following proposition

Proposition 4. *If X is not constant, $X_n \Rightarrow X$ and $a_n > 0, b_n$ are such that $a_n X_n + b_n \Rightarrow Y$, then there exists $a = \lim_n a_n$, $b = \lim_n b_n$ and, therefore, $Y = aX + b$ (in distribution).*

Proof: If Y is constant then, from $a_n X_n + b_n - Y \Rightarrow 0$ (see 1. above) it follows, via Corollary 2, that $\lim_n a_n = 0$ and $\lim_n b_n = Y$.

If Y is not constant we apply Lemma 5 with F_n , F , and G the distribution functions of X_n , X and Y respectively, and with $u_n = 1$, $v_n = 0$. \square

Remark: Taking $X_n = 1$ (the constant function), $a_n = n$, and $b_n = -n$, we see that the given restriction on X (to be non constant) is necessary.

We finish this appendix with a lemma covering the asymptotically degenerate case.

Lemma 7. *If Y is constant, $Y_k \Rightarrow Y$, and $Y_k + c_k \Rightarrow Z$ then $c = \lim_k c_k$ exists and $Z = Y + c$.*

Proof: Use $Y + c_k = (Y - Y_k) + Y_k + c_k \Rightarrow Z$ by 2. and 3. above. \square

Acknowledgements

The author wants to thank Magda Peligrad for proposing the question of this note and for our discussions related to it. Valuable comments and suggestions are also due to Dalibor Volný. The author was supported by the NSF Grant DMS-1208237.

References

- [1] Barrera, D. and Peligrad, M. (2014) Quenched Limit Theorems for Fourier Transforms and Periodogram. To appear in *Bernoulli*.
- [2] Billingsley, P. (1995). Probability and Measure, 3-rd edition. *Wiley, New York*.
- [3] Carleson, L. (1966). On convergence and growth of partial sums of Fourier series. *Acta Math.* **116**. 135-157.
- [4] Cuny, C., Merlevede, F. and M. Peligrad (2013). Law of the iterated logarithm for the periodogram. *Stoch. Proc. Appl.* **123** 4065-4089.
- [5] Doob, J. Stochastic Processes. Wiley, 1953.
- [6] Peligrad, M. and W. B. Wu (2010). Central limit theorem for Fourier transforms of stationary processes. *Ann. Probab.* **38** 2009-2022.
- [7] Petersen, K. (1989). Ergodic theory. Corrected reprint of the 1983 original. *Cambridge Studies in Advanced Mathematics* **2**. Cambridge University Press.
- [8] Volný, D. and M. Woodroffe. (2010). An example of non-quenched convergence in the conditional central limit theorem for partial sums of a linear process. *Dependence in analysis, probability and number theory (The Phillip memorial volume)*, Kendrick Press. 317-323.